MTH 265 Notes. Week 2 Monday + Wednesday.

(A) The Divergence Test (DT)

Let $\sum a_n$ be a series.

1. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is divergent.
2. If $\lim_{n\to\infty} a_n = 0$, the test is inconclusive.

Examples. ① $\sum_{n=1}^{\infty} (-1)^{n+1}$; Since $\lim_{n\to\infty} (-1)^{n+1} \neq 0$, $\sum (-1)^{n+1}$ is divergent.

② $\sum_{n=1}^{\infty} \frac{1}{n}$; Since $\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$, the DT is inconclusive.

As a sidenote, $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent since it's a p-series with $p=1$.

③ $\sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n$; Since $\lim_{n\to\infty} \left(\frac{1}{8}\right)^n = 0$, the DT is inconclusive.

As a sidenote, $\sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n$ is convergent since it's a geometric series with $r = \frac{1}{8}$;

④ $\sum_{n=1}^{\infty} \left(\ln(n) - \ln(n+1)\right)$; Since $\lim_{n\to\infty} \left[\ln(n) - \ln(n+1)\right] = \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right) = \ln\left(\lim_{n\to\infty} \frac{n}{n+1}\right) = \ln(1) = 0$.

the DT is inconclusive.

As a sidenote, $\sum [\ln(n) - \ln(n+1)]$ is a telescoping series but since $\lim_{n\to\infty} \ln(n) = \infty$, the series diverges.

⑤ $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$; Since $\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{1}{n}\right) = \cos(0) = 1 \neq 0$,

$\sum \cos\left(\frac{1}{n}\right)$ is divergent by DT.

⑥ $\sum_{n=1}^{\infty} \frac{1}{n^2}$; Since $\lim_{n\to\infty} \frac{1}{n^2} = 0$, DT is inconclusive.

⑦ $\sum_{n=1}^{\infty} \left(2 + \frac{1}{e^n}\right)$ is divergent by DT since $\lim_{n\to\infty} \left(2 + \frac{1}{e^n}\right) = \lim_{n\to\infty} (2) = \infty$;

(B) Let $\sum a_n$ be convergent and $\sum b_n$ be divergent. Then, $\sum (a_n + b_n)$ is divergent.

This is an application of the result (convergent + div = div) on sequences to the sequence of partial sums.

Examples: ① $\sum_{n=1}^{\infty} \left(\frac{n+1}{n^2}\right)$; By PFD: $\frac{n+1}{n^2} = \frac{A}{n} + \frac{B}{n^2}$; $n+1 = A(n) + B$; $A=1$, $B=1$;

So, $\sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{1}{n^2}\right)$ is divergent since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the p-series test.

(C) Correction.

The geometric series formula: if $|r| < 1$ and $r \neq 0$: $\sum_{n=n_0}^{\infty} ar^n = \frac{ar^{n_0}}{1-r}$

This is added since if $r=0$, $\sum ar^n = \sum (0) = 0$.

The derivation of the formula $\sum_{n=n_0}^{N} ar^n = a\left(\frac{r^{n_0} - r^{N+1}}{1-r}\right)$ assumes $r \neq 0$.

Ⓓ Let $\sum a_n = A$ and $\sum b_n = B$ be convergent series. Let $k \in \mathbb{R}$.

① $\sum (ka_n)$ is convergent with $\sum (ka_n) = k \sum a_n = kA$;

② $\sum (a_n + b_n)$ is convergent with $\sum (a_n + b_n) = \sum (a_n) + \sum (b_n) = A + B$;

Examples: ① $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2^n} + \frac{1}{3^n}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{3^n}\right) = \frac{(1)(\frac{1}{2})^1}{1-\frac{1}{2}} + \frac{(1)(\frac{1}{3})^1}{1-\frac{1}{3}} = \frac{\frac{1}{2}}{\frac{1}{2}} + \frac{\frac{1}{3}}{\frac{2}{3}} = 1 + \frac{1}{2} = \frac{3}{2}$;

② $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3^n + 2^n}{6^n}\right) = \sum_{n=1}^{\infty}\left(\frac{3^n}{6^n} + \frac{2^n}{6^n}\right) = \sum_{n=1}^{\infty}\left(\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n\right) = \frac{3}{2}$;

③ Find c such that $\displaystyle\sum_{n=0}^{\infty}\left(\frac{3^n + c^n}{6^n}\right) = 2$;

Assume c such that $\left|\frac{c}{6}\right| < 1$:

$$\sum_{n=0}^{\infty}\left(\frac{3^n + c^n}{6^n}\right) = \sum_{n=0}^{\infty}\left(\left(\frac{1}{3}\right)^n + \left(\frac{c}{6}\right)^n\right) = \frac{(1)(\frac{1}{3})^0}{1-\frac{1}{3}} + \frac{(1)(\frac{c}{6})^0}{1-\frac{c}{6}} = \frac{1}{\frac{2}{3}} + \frac{1}{1-\frac{c}{6}}\left(\frac{6}{6}\right)$$

$= \frac{3}{2} + \frac{6}{6-c} = 2$; $3(6-c) + 6(2) = 2(2)(6-c)$;

$18 - 3c + 12 = 24 - 4c$; $-3c + 4c = 24 - 18 - 12$; $c = -6$;

This contradicts $\left|\frac{c}{6}\right| = \left|\frac{-6}{6}\right| = |-1| = 1 \not< 1$.

Assume $c = 0$: $\displaystyle\sum_{n=0}^{\infty}\left(\frac{3^n + (0)^n}{6^n}\right) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1(\frac{1}{2})^0}{1-\frac{1}{2}} = 2$;

Therefore, $c = 0$.

Ⓔ Rearrangement + Regrouping.

Definition. Let $\sum a_n$ be a series. Then,

① If $\sum |a_n|$ is convergent, then $\sum a_n$ is absolutely convergent.

② If $\sum |a_n|$ is divergent but $\sum a_n$ is convergent, then $\sum a_n$ is conditionally convergent.

Examples: ① $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n$ is absolutely convergent.

② $\displaystyle\sum_{n=0}^{\infty}\left(-\frac{1}{9}\right)^n$ is absolutely convergent.

③ $\displaystyle\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n}\right)$ is conditionally conv. as we will see later when we talk about alternating series.

Ⓕ Let $\sum a_n$ be a series and let $\sum b_n$ be a rearrangement or regrouping of the series. Then, $\sum a_n$ is absolutely conv. if and only if $\sum b_n$ is convergent with $\sum a_n = \sum b_n$.

That is, if $\sum a_n$ is not absolutely conv., then $\sum b_n$ may be divergent or result in a diff. sum.

Example: $a_n = (-1)^n$. $\displaystyle\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty}(-1)^n = \text{DNE}$.

Regroup. Let $b_n = a_{2n} + a_{2n+1}$ for $n \in \mathbb{Z}_{\geq 0}$. Then, $b_n = a_{2n} + a_{2n+1} = (-1)^{2n} + (-1)^{2n+1} = (1) + (-1) = 0$;

Observe that $\displaystyle\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty}(0) = 0$; Key Point: Series are not "commutative" or "associative"

⑥ The Integral Test (IT)

Let $f(x)$ be positive, decreasing, and continuous for $x \geq n_0$.

Then, $\sum_{n=n_0}^{\infty} f(n)$ converges if and only if $\int_{n_0}^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_{n_0}^{b} f(x)\,dx$ converges.

EXAMPLES. ① Determine the convergence of $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$;

Let $f(x) = \frac{1}{x}$: We need to check the conditions are satisfied.

① For $x \geq 1$, $f(x) = \frac{1}{x}$ is positive.

② $f(x) = \frac{1}{x}$ is a rational function and they are continuous on their domain.

$\therefore f(x)$ is continuous on $x \neq 0$. $\therefore f(x)$ is continuous for $x \geq 1$.

③ $f'(x) = (-1)(x)^{-2} = -\frac{1}{x^2}$; for $x \geq 1$, $f'(x)$ is negative. $\therefore f(x)$ is decreasing.

Then, $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln|x|\right]_1^{b} = \lim_{b\to\infty}\left[\ln(b) - \ln(1)\right] = \infty$;

$\therefore \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by IT.

② The p-series test is simply the Integral Test on $f(x) = x^{-p}$.

③ $\sum_{k=1}^{\infty} ke^{-3k^2}$; let $f(x) = xe^{-3x^2}$; Restrict $x \in [1, \infty)$.

Check the conditions:

① $f(x)$ is continuous on $\mathbb{R}$.

② For $x \in [1,\infty)$: x is positive; e^{-3x^2} is always positive; $\therefore f(x)$ is positive.

③ $f'(x) = x(-3x)e^{-3x^2} + e^{-3x^2} = e^{-3x^2}(-6x^2+1)$;

Since $x \geq 1$, $6x^2 \geq 6 > 1$; $1 - 6x^2 < 0$; Since e^{-3x^2} is always positive, $f'(x)$ is negative and $f(x)$ is decreasing on $x \in [1,\infty)$.

Then, $\int_1^{\infty} xe^{-3x^2}\,dx = \lim_{b\to\infty} \int_1^{b} xe^{-3x^2}\,dx = \lim_{b\to\infty} \int_{-3}^{-3b^2} -\frac{1}{6}e^{u}\,du = \lim_{b\to\infty}\left[-\frac{1}{6}e^{u}\right]_{-3}^{-3b^2}$

$\left[\begin{array}{l} u = -3x^2;\ du = -6x\,dx;\ -\frac{1}{6}du = x\,dx; \\ x=b:\ u = -3b^2;\ x=1:\ u=-3; \end{array}\right]$

$= -\frac{1}{6}\lim_{b\to\infty}\left[e^{-3b^2} - e^{-3}\right] = -\frac{1}{6}\left(0 - e^{-3}\right) < \infty.$

By the Integral Test, $\sum_{k=1}^{\infty} ke^{-3k^2}$ is convergent.